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TWO PARABOLIC GENERATOR KLEINIAN GROUPS

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ABSTRACT

This paper is concerned with torsion-free Kleinian groups that are generated by two parabolic transformations. Our main result is that every such group of the second kind is geometrically finite; this is in response to a question raised by Riley. We also show that in the natural one (complex) dimensional setting, the space of torsion-free Kleinian groups of the second kind is path-connected.

The primary purpose of this note is to show that every torsion free Kleinian group of the second kind generated by two parabolic transformations is geometrically finite; this is in response to a question raised by R. Riley (oral communication). We also show that the conclusion fails if we drop the hypothesis that the group be of the second kind.

We consider only torsion-free groups in this paper; from here on, without further mention, all groups are assumed to be torsion-free.

THEOREM 1. *Let G be a Kleinian group of the second kind generated by two parabolic transformations. Then G is geometrically finite.*

It is well known that the above theorem fails if we drop the assumption that G be of the second kind (see Greenberg [G]). For the convenience of the reader, we include a proof.

THEOREM 2. *There is a Kleinian group G of the first kind generated by two parabolic transformations that is not geometrically finite.*

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Throughout this paper, G is a Kleinian group of the second kind; that is, if we regard G as acting on the extended complex plane, $\hat{\mathbf{C}}$, then the set of discontinuity, $\Omega = \Omega(G) \neq \emptyset$. Also, throughout this paper, G is generated by two parabolic transformations, A and B; we write $G = \langle A, B \rangle$.

We normalize so that $A(z) = z + 1$, and so that $B(z)$ has its fixed point at 0; then $B(z) = z/(\sigma z + 1)$. When we need to emphasize the dependence on σ, we will write $B = B_\sigma$ and $G = G_\sigma$.

THEOREM 3. *The set of points σ for which G_σ is Kleinian of the second kind is path connected.*

The basic idea for the proof of Theorem 3 is due to Marden (unpublished); see also Abikoff [A], and Maskit [M3].

1. Basics

We will in general regard G as acting on hyperbolic 3-space, $\mathbf{H}^3 = \{(z, t) \mid z \in \mathbf{C}, t > 0\}$, and we will regard $\hat{\mathbf{C}}$ as the sphere at infinity of $\mathbf{H}^3$. Let $M = (\mathbf{H}^3 \cup \Omega(G))/G$, and let $p: \mathbf{H}^3 \cup \Omega \rightarrow M$ be the natural projection. By Ahlfors' finiteness theorem, ∂M has finitely many components, and each of them is (conformally) a closed Riemann surface from which a finite number of points has been removed.

A *deformation* of G is a quasiconformal mapping $w: \hat{\mathbf{C}} \rightarrow \hat{\mathbf{C}}$, where w fixes the triple of points $(0, 1, \infty)$, and w conjugates G into another Kleinian group, $\tilde{G}$. Then $w \circ A \circ w^{-1} = A$, and $w \circ B \circ w^{-1} = \tilde{B}$ is again of the same form; i.e., $\tilde{B}(z) = z/(\tilde{\sigma} z + 1)$. It is an easy consequence of the Ahlfors–Bers Riemann mapping theorem [A–B] that $\tilde{\sigma}$ depends holomorphically on w. Two deformations are *equivalent* if they induce the same isomorphism; i.e., if they both conjugate B onto the same transformation. The space of equivalence classes of deformations is the *deformation space*, $T = T(G)$. It is clear that the mapping $w \rightarrow \tilde{\sigma}$ induces a holomorphic injection from $T(G)$ into the σ plane.

It was shown by Sullivan [Su] that every deformation of G is equivalent to one supported on the regular set (that is, every point of T can be represented by a deformation that is conformal on the limit set), and it was shown by Bers [B1], Kra [K], and Maskit [M1] that the deformation space supported on the regular set is a holomorphic product, $T = T_1 \times \cdots \times T_k$, of complex manifolds, where the factors are obtained as follows. Write $\Omega(G)/G = S_1 \cup \cdots \cup S_k$ as a disjoint union of Riemann surfaces. Then there is a discrete subgroup of the Teichmüller Modular group acting on the Teichmüller space of S_j so that T_j is the Teichmüller space $T(S_j)$ factored by this discrete subgroup. It follows that if S_j is a Riemann surface of genus g_j with n_j punctures (by Ahlfors finiteness theorem, this is the only possibility), then T_j has dimension $3g_j - 3 + n_j$.

Since T has dimension 1, $\Sigma(3g_j - 3 + n_j) = 1$. The only possibilities for this to occur are (i) some S_j is a torus with one puncture, and all the other S_j are 3-times punctured spheres; or (ii) some S_j is a 4-times punctured sphere, and all the others are 3-times punctured spheres; or (iii) every S_j is a 3-times punctured sphere.

It was shown by Ratcliffe [R] that every two generator Kleinian group G is either (i) elementary or (ii) of cofinite volume or (iii) free. In the first two cases, G is necessarily geometrically finite; from here on we consider only the case that G is free.

2. Cores

It was shown by Scott [S] that M has a *core*, that is, there is a compact submanifold $X \subset M$ so that the inclusion from X to M is a homotopy equivalence. This was refined by McCullough [Mc], who showed that X can be chosen so that if Y is any compact submanifold of ∂M, then $X \cap \partial M = Y$ (see also Kulkarni and Shalen [K–S]).

We need to make some modifications in our manifold M, and to choose the compact submanifold Y of ∂M with some care. We give two equivalent constructions.

First construction. For each sufficiently small $\mu > 0$, let $M_0(\mu)$ be the thick part of M. That is, let $d(\cdot, \cdot)$ denote hyperbolic distance, and let

$$T(\mu) = \{x \in \mathbf{H}^3 \mid d(x, g(x)) < \mu \text{ for some } g \in G\};$$

then let $M_0(\mu) = ((\mathbf{H}^3 \cup \Omega(G)) - T(\mu))/G$. By Margulis's theorem, there is a number μ_0 so that for all $\mu < \mu_0$, ∂M_0 consists of a finite number of disjoint tori and cylinders. We choose some $\mu < \mu_0$, and let $M_0 = M_0(\mu)$.

There are two kinds of tori in ∂M_0. The *parabolic tori* are incompressible, and their fundamental groups, as subgroups of G, are purely parabolic. The *hyperbolic tori* are compressible; the images of their fundamental groups in G are hyperbolic (including loxodromic) cyclic. Each hyperbolic torus on ∂M_0 divides M into two components; one of these, which is disjoint from M_0, is a solid torus. We adjoin these closed solid tori to M_0, to obtain a new submanifold M_1, which is a deformation retract of M.

Note that ∂M_1 has three kinds of components. These are (i) the parabolic incompressibly embedded tori, (ii) the components of ∂M, and (iii) some properly embedded infinite cylinders, henceforth called *infinite cylinders*.

If C is an infinite cylinder, then $\pi_1(C)$ is a (cyclic) maximal parabolic subgroup of G. Conversely, if J is a maximal parabolic subgroup of G, where J has rank 1, then, after making appropriate choices of base points, there is an infinite cylinder C, so that $J = \pi_1(C)$.

Write $\partial M = N_1 + \cdots + N_n$, where each N_j is a connected component. Let p be a puncture on some N_j. Then there is a simple loop w surrounding p; i.e., w bounds a punctured disc $D \subset N_j$, where the puncture is p.

Since p is a puncture, the homotopy class of w is parabolic; in fact it is a generator of a maximal parabolic subgroup of G of rank 1. Hence there is an annulus A in M, where one boundary component of A is w, and the other is an essential loop on some infinite cylinder C. The annulus A divides M into two components; one of these, call it B, contains D and has infinite cyclic fundamental group. We delete the interior of B from M_1; this has the effect of attaching the component N_j to the infinite cyclinder C, and replaces the puncture p by an infinite end of C, but has no effect on homotopy; i.e., the manifold M_1' obtained by deleting the interior of B is still a deformation retract of M.

We repeat the above operation for every puncture (of course, keeping the annuli disjoint), and obtain a new submanifold M_2. Again, M_2 is a deformation retract of M. If ∂M_2 is not compact, then every non-compact end can be identified with an infinite end of an infinite cylinder. We truncate each of these infinite cylinders as necessary, and so obtain a compact submanifold N_0 of ∂M_2. Let X be the relative core of M_2, where $X \cap \partial M_2 = N_0$. Of course, X is also a core for M.

Second construction. It is well known that there is a G-invariant collection of closed disjoint horoballs in $\mathbf{H}^3$, where each of these horoballs is precisely invariant under the maximal parabolic group with fixed point at its vertex (the vertex of a horoball is the point of tangency with the sphere at infinity). Let T_1 be the complement of the union of the interiors of these horoballs, and let $M_1 = T_1/G$. Note that the boundary of M_1 is as described above. For each puncture on ∂M, there is an invariant collection of closed disjoint circular discs, with interior in $\Omega(G)$, where each of these discs is precisely invariant under a maximal parabolic subgroup of rank 1. It is also well known that we can choose the discs corresponding to one puncture to be disjoint from the discs corresponding to another, except perhaps at the parabolic fixed points (see [M4, p. 121]).

Each of the circular discs described above is the boundary at infinity of a hyperbolic half-space. Let T_2 be the complement, in T_1, of the union of the interiors of all these half-spaces, and let $M_2 = T_2/G$. Then ∂M_2 consists of closed surfaces of positive genus, and a finite number of open surfaces, where the non-compact ends of the open surfaces are the projections of the boundaries of those horoballs B satisfying the following. The stabilizer of B in G is cyclic (necessarily parabolic), and B intersects fewer than two of these half-spaces. If B intersects none of these half-spaces, then $\partial M_2 \cap (B/\mathrm{Stab}(B))$ is an infinite cylinder, while if B intersects one of these half-spaces, then $\partial M_2 \cap (B/\mathrm{Stab}(B))$ is an infinite cylinder where one end has been peeled off, so it has one compact end and one open end. In either case, we choose some compact sub-cylinder, and peel off the infinite ends. As above we are left with a compact sub-manifold N_0 of ∂M_2 (the complement of N_0 in ∂M_2 is exactly the infinite ends of the cylinders we have peeled off). We choose a core X for M_2, which is then necessarily a core for M, so that $X \cap \partial M_2 = N_0$.

It is easy to check that the two constructions given above are equivalent, that is, the core X, and the subsurface of the boundary, N_0, are essentially the same in both constructions.

3. Boundary considerations I; punctured tori

Standard applications of the loop theorem and Dehn's lemma show that since X is an aspherical compact 3-manifold, where $\pi_1(X)$ is free on two generators, X is a handlebody of genus 2. Embedded in ∂X we have ∂M, with its punctures truncated, and cylinders coming from the parabolic elements of G. Since there is exactly one such cylinder for each conjugacy class of maximal parabolic subgroups, the essential loops of these cylinders are homotopically distinct in X, and so also in ∂X. It follows that there are at most three such cylinders; that is, in addition to the two generators A and B, there is at most one additional parabolic element of G, not conjugate to any power of A or B. Call these cylinders C_1, C_2, and C_3, where C_3 might be empty.

We next suppose there is a punctured torus T' on ∂M. We observe that T' is incompressibly embedded in M, for otherwise, using the loop theorem, we would have that the punctured disc near the boundary of T' would be compressible.

We delete a neighborhood of the puncture on T', and obtain a compact torus with one boundary component, $T \subset \partial M_1$. We can also regard T as lying in ∂M_2, hence in ∂X. Since ∂X is a closed orientable surface of genus 2, the complement, S, of T is also a torus with one hole. By construction, T is disjoint from C_1, C_2, and C_3, so they all lie in S. We have shown that one of the C_j, call it C', divides ∂X. Since the essential loop in C' is a commutator in $\pi_1(\partial X)$, it is also a commutator in $\pi_1(X) = G$. Of course, A and B are not commutators, so $C' = C_3$. Then the other two cylinders, C_1 and C_2, must be disjoint and non-dividing on S; so their essential loops are freely homotopic. This implies that the free generators A and B are conjugate, which of couse they are not. We have shown that there is no punctured torus on ∂M.

4. Boundary considerations II; 4-times punctured spheres

We next assume that there is a 4-times punctured sphere, T, on ∂M. Since all four punctures correspond to parabolics in G, and G contains at most three non-conjugate parabolics, T is compressible. Then by the loop theorem, or the planarity theorem, there is a simple loop w on T, where w divides two of the punctures on T from the other two, and w is homotopically trivial in M. Let $H \subset G$ be the stabilizer of a component Δ of Ω lying over T. Then H is a free product, using combination theorems, of two parabolic cyclic groups. It is well known that for such a group, $\Delta = \Omega$ (see [M4, p. 149ff] for the proof that if G_1 and G_2 both have only one component, then so does their free product). Since $\Delta = \Omega$, $H = G$, and G, being the free product of two geometrically finite groups, is also geometrically finite [M4, p. 149ff].

We have shown that if $\Omega(G)/G$ contains a 4-times punctured sphere, then G is geometrically finite, and we have also shown that in this case $\Omega(G)$ is connected, from which it follows that ∂M is connected.

5. Boundary considerations III; 3-times punctured spheres

We now take up the last case, that ∂M consists only of 3-times punctured spheres. Since there are at most three distinct parabolics in G, ∂M contains at most 6 punctures, so there are at most two of these components.

If ∂M contains two 3-times punctured spheres, then each parabolic in G represents two punctures on ∂M, so in the terminology of [M3], G is free and evenly cusped, hence geometrically finite. We have essentially reproved this result here; see Swarup [Sw]. That is, since each parabolic represents two punctures on ∂M, ∂M_2 is a closed surface of genus 2; hence by Marden's criterion [Ma], G is geometrically finite.

If ∂M contains only one 3-times punctured sphere, T, then $\partial X - \partial M_2$ is a sphere with three holes, call it S. Let $\tilde{S}$ be some connected component of

$p^{-1}(S)$. Then $F = \mathrm{Stab}(\tilde{S})$ is generated by two parabolics whose product is again parabolic. Standard elementary computations in PSL(2, C) show that F is Fuchsian; i.e., $\Lambda(F)$, the limit set of F, is a circle. One of the circular discs bounded by $\Lambda(F)$ contains all the lifts of T; the other is necessarily precisely invariant under F in G. It follows that there is a component of ∂M parallel to S, contrary to our assumption.

We have shown that ∂M cannot consist of only one 3-times punctured sphere, and that if ∂M contains two 3-times punctured spheres, then G is geometrically finite. This concludes the proof of Theorem 1.

6. Proof of Theorem 3

We saw above that if G_σ is free, then it is geometrically finite, and it was shown in [M3] that every free geometrically finite Kleinian group lies on the boundary of the appropriate space of Schottky groups. The proof below is closely related to that given in [M3].

We start with the well known fact that the set of all σ for which G_σ represents a 4-times punctured sphere is connected; that is, we can go from any one such group to any other by a path of quasiconformal deformations. To see this, start with some fixed σ_0, and observe that by lifting quasiconformal mappings, every homotopy class of mappings from $\Omega(G_{\sigma_0})/G_{\sigma_0}$ onto a 4-times punctured sphere can be represented by a quasiconformal deformation of G_{σ_0}. Then the standard trick of multiplying the dilitation by a positive real variable, and solving the corresponding Beltrami equation [A–B], shows that it can be connected to the identity via a path of quasiconformal homeomorphisms. We have already observed that every group in our space representing a 4-times punctured sphere is geometrically finite, and has connected ordinary set. It now follows from the uniqueness theorem for geometrically finite function groups (see [M2]) that the space of geometrically finite function groups representing 4-times punctured spheres is connected.

Now let $G = G_\sigma$ be of the second kind, where $\Omega(G)/G$ is not a 4-times punctured sphere. Then, as we saw above, $\Omega(G)/G$ is the union of two thrice punctured spheres; call them T_1 and T_2. Then there is some primitive parabolic element $C \in G$, so that every parabolic element is conjugate in G to some power of either A, B, or C, and C is not conjugate to any power of A or B.

We next construct a path of *partial cores based at* C as follows. We first renormalize G so that $C(z) = z + 1$, and so that the origin is an interior point of $\Omega(G)$.

For each sufficiently large $s \in \mathbf{R}$, the set $\{(z,t) \mid |\mathrm{Im}(z)| > s\} \cup \{(z,t) \mid t > s\} \cup \{\infty\}$ is precisely invariant under $\langle C \rangle$ in G (see [M3]). Let $T(s)$ be the complement of the union of the translates of this set. Then $T(s)$ is closed, and G-invariant. Further, $\partial T(s)/G$ is ∂M, where neighborhoods of the punctures corresponding to C have been cut out and replaced by Marden pairing tubes (see [Ma]). Since there are only finitely many corners, it is clear how to define a complex structure on $\partial T(s)/G$, so that it is a (connected) Riemann surface. This Riemann surface is a 4-times punctured sphere, where the punctures are paired — the paring is given by the fact that A and B each represent two punctures.

Given the Riemann surface, $\partial T(s)/G$, there is a unique Kleinian group $H = H(s)$, so that $\Omega(H)/H = \partial T(s)/G$, where the regular covering $\Omega \to \Omega/H$ is topologically and conformally equivalent to the covering $\partial T(s) \to \partial T(s)/G$ [M2]. An equivalent statement is that for each sufficiently large s there is a conformal homeomorphism f_s, defined in $\Omega(G) \cap \partial T(s)$, where f_s conjugates G onto the Kleinian group $H(s)$, and $f_s \circ A \circ f_s^{-1}$ and $f_s \circ B \circ f_s^{-1}$ are both parabolic. Of course, $f_s \circ C \circ f_s^{-1}$ is not parabolic but loxodromic.

We have already renormalized G once; we now do so again, so that ∞ is an interior point of $\Omega(G)$. As above, this renormalization is independent of s. We also normalize each f_s so that $f_s(\infty) = \infty$, and so that near ∞, $f_s(z) = z + O(1/|z|)$. Let Δ_0 be the component of G containing ∞.

Let s_k be any sequence of real numbers tending to infinity. By the Koebe one-quarter theorem, the functions $f_{s_k} = f_k$ are uniformly bounded and holomorphic in Δ_0 outside a fixed neighborhood of ∞; hence we can find a convergent subsequence. For any component other than Δ_0, the f_k are holomorphic and uniformly bounded, hence we can find a subsequence, which we again call by the same name, so that $f_k(z)$ converges uniformly on compact subsets of Ω to some (holomorphic) function f. Since each f_k is univalent, f is either univalent or constant on each component. It follows at once from the normalization that f is not constant on Δ_0, hence $f \mid \Delta_0$ is univalent. It follows that for $g \in \mathrm{Stab}(\Delta_0)$, the stability subgroup of Δ_0, $f_k \circ g \circ f_k^{-1}$ converges to the Möbius transformation, $f \circ g \circ f^{-1}$.

It is easy to see that if f is the constant z on the component Δ, then for every $g \in \mathrm{Stab}(\Delta)$, $f_k \circ g \circ f_k^{-1}$ converges uniformly to the constant z on compact subsets of Δ.

Of course, for fixed g, the sequence of Möbius transformations $f_k \circ g \circ f_k^{-1}$ either converges to a Möbius transformation or to a constant. Hence if Δ_1 and Δ_2 are both components of G with a common parabolic fixed point on their boundary (i.e., they are both stabilized by the same parabolic $g \in G$), then $f \mid \Delta_1$ and $f \mid \Delta_2$ are either both univalent or both the same constant. Since $f \mid \Delta_0$ is univalent, and we can reach every component of G by a path that goes from component to component by passing through parabolic cusps, f is univalent on every component of G. Since each component of G is a circular disc, f is in fact a Möbius transformation on each component.

Since f is locally univalent, and each f_k is globally univalent, f is globally univalent. Of course, for each $g \in G$, $f_k \circ g \circ f_k^{-1}$ converges to $f \circ g \circ f^{-1}$, a Möbius transformation. Hence f conjugates G onto the Kleinian group fGf^{-1}. Now by the Marden isomorphism theorem [Ma] f is a global Möbius transformation. It then follows from the normalization near ∞ that f is the identity. It follows that for every $g \in G$, $f_k \circ g \circ f_k^{-1} \to g$.

We now return to our original normalization, and simultaneously renormalize the f_s, so that $f_s(z) \to z$ uniformly on compact subsets of $\Omega(G)$. Given the point σ_0, we define the path $\sigma(s)$ in our parameter space by $f_s \circ B \circ f_s^{-1}(z) = z/(\sigma(s)z + 1)$. Then for any sufficiently large s, $\sigma(s)$ corresponds to a (geometrically finite) Kleinian group representing a 4-times punctured sphere, and for every sequence $s_k \to \infty$, there is a subsequence, which we call by the same name, so that $\sigma(s_k) \to \sigma_0$. This concludes the proof of Theorem 3.

7. Proof of Theorem 2

The classical Shimizu–Leutbecher lemma says that if $G(\sigma)$ is discrete, then $|\sigma| \geqq 1$. If $|\sigma| > 4$, then the isometric circles of B and B^{-1} are both completely contained within the strip $|\mathrm{Re}(z)| < \frac{1}{2}$, so $G(\sigma)$ is the free product, in the sense of combination theorems, of $\langle A \rangle$ and $\langle B \rangle$; in this case, $\Omega(G)/G$ is a 4-times punctured sphere.

For each element g in the (abstract) free group generated by A and B, the set of points σ for which $G(\sigma)$ is either parabolic or the identity is the set of roots of the polynomial, $\mathrm{trace}^2(g) = 4$. Hence, for each g, the set of such σ is finite. Since this abstract group is countable, for almost every ray through the origin, the group $G(\sigma)$, with σ on that ray, is free, and contains no parabolics other than conjugates of powers of A and B. We choose such a ray, $\arg(\sigma) = \theta_0$, and write $\sigma = \rho e^{i\theta_0}$. We start with ρ sufficiently large, so that $G(\sigma)$ represents a 4-times punctured sphere, and let ρ decrease. Since the set of σ for which $G(\sigma)$ represents a 4-times punctured sphere is open, there is a first point ρ_0, so that $G_0 = G(\sigma_0) = G(\rho_0 e^{i\theta_0})$ does not represent a 4-times punctured sphere.

We first remark that G_0 is discrete. This follows from the fact that $\rho \geqq 1$. For every $\rho > \rho_0$, the closed horoball of unit height is precisely invariant under the stabilizer of ∞ in $G(\rho e^{i\theta_0})$; hence the open horoball of unit height is precisely invariant under the stabilizer of ∞ in G_0.

We already know that G_0 is free. Hence, by Theorem 1, if it were of the second kind, it would be geometrically finite. In this case, as we saw above, since G_0 does not represent a 4-times punctured sphere, it would have to represent two 3-times punctured spheres, so the total number of conjugacy classes of maximal parabolic subgroups would be three, which contradicts our choice of θ_0. We conclude that G_0 is of the first kind.

It is well known that a free geometrically finite Kleinian group is necessarily of the second kind. To see this, observe that if G were of the first kind, then every pure parabolic subgroup would have rank 2 (this follows from the Beardon–Maskit criterion for geometric finiteness [M4, p. 128]), so G would be purely loxodromic, hence cocompact. As remarked above, a compact irreducible 3-manifold with free fundamental group is a handlebody, and hence has boundary. We have shown that G_0 is not geometrically finite.

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